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The $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$ Boundedness for Some Rough Operators*

Yong Ding and Shanzhen Lu

Department of Mathematics, Beijing Normal University, Beijing 100875, The People's Republic of China

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In this paper, the authors study $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$ boundedness of $M_{\Omega,\alpha}(\mathbf{f})$ and $T_{\Omega,\alpha}(\mathbf{f})$, which are the k-sublinear fractional maximal operator and the k-linear fractional integral operator with rough kernel, respectively.

1. INTRODUCTION

Suppose that Ω is homogeneous of degree zero and $\Omega \in L^s(S^{n-1})$ $(s > 1)$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. Moreover, $k \geq 2$ will denote an integer, θ_j $(j = 1, 2, \ldots, k)$ will be fixed, distinct, and nonzero real numbers, and $0 < \alpha < n$. It is said that p is the harmonic mean of $p_1, p_2, \ldots, p_k > 1$ if $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_k$. We denote $\mathbf{f} = (f_1, f_2, \ldots, f_k)$. If $f_j \in L^{p_j}(\mathbb{R}^n)$, $j = 1, 2, \ldots, k$, then we say that $\mathbf{f} \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$. In the following we define the k-sublinear fractional maximal operator by

$$M_{\Omega,\alpha}(\mathbf{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)|\,|f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)|\,dy$$

and the k-linear fractional integral operator by

$$T_{\Omega,\alpha}(\mathbf{f})(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x - \theta_1 y) \cdots f_k(x - \theta_k y)\,dy.$$

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It is worth pointing out that the study of $L^{p_1} \times L^{p_2} \times \cdot \times L^{p_k}$ boundedness for $T_{\Omega,\alpha}(\mathbf{f})$ is significant. In fact, when $\alpha = 0$, almost nothing is known about the boundedness of the corresponding singular integral operators, or even the maximal function, when $p = 1$. Thus considering an $\alpha > 0$ is a natural simplification step toward the positive solution of the problem for the singular integral operators. When $k = 1$ and $\theta_1 = 1$, the weak boundedness, L^p-boundedness, and the weighted L^p-boundedness of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ have been studied. For example, Muckenhoupt and Wheeden [9] set up the weighted L^p-boundedness of $T_{\Omega,\alpha}$ for power weights. Chanillo, Watson, and Wheeden [4] proved that $T_{\Omega,\alpha}$ is of weak type $(1, n/(n-\alpha))$. In [6], we studied the weighted L^p-boundedness of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ for general weights. On the other hand, Grafakos [7] studied the $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$ boundedness for $T_{\Omega,\alpha}(\mathbf{f})$ in the case $\Omega \equiv 1$ and $k \geq 2$. The aim of this paper is to obtain more general results than that of [7] for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$, where $\Omega \in L^s(S^{n-1})$, $s > n/(n-\alpha)$. Let us now formulate our results as follows.

THEOREM 1. *Suppose that* $0 < \alpha < n, \Omega \in L^s(S^{n-1}), s > n/(n-\alpha)$. *Let* p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$ *and* q *satisfy* $1/q = 1/p - \alpha/n$. *Then* $T_{\Omega,\alpha}(\mathbf{f})$ *is bounded operator from* $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *for* $n/(n+\alpha) \leq p < n/\alpha$ (*equivalently* $1 \leq q < \infty$).

THEOREM 2. *If* α, Ω, s, p, *and* q *are as in Theorem* 1, *then* $M_{\Omega,\alpha}(\mathbf{f})$ *is a bounded operator from* $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *for* $n/(n+\alpha) \leq p \leq n/\alpha$ (*equivalently* $1 \leq q \leq \infty$).

Theorem 2 contains the boundedness of $M_{\Omega,\alpha}(\mathbf{f})$ in the endpoint case $p = n/\alpha$, but the corresponding result for $T_{\Omega,\alpha}(\mathbf{f})$ in this case does not hold. However, the following theorem can be regarded as the substitute of the boundedness for $T_{\Omega,\alpha}(\mathbf{f})$ in the endpoint case $p = n/\alpha$.

THEOREM 3. *Suppose that* $0 < \alpha < n, \Omega \in L^s(S^{n-1})$, $s \geq n/(n-\alpha)$. *Let* $p = n/\alpha$ *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$. *Let* $B = \{x \in \mathbb{R}^n : |x| < R\}$ *and* $f_j \in L^{p_j}(B)\ (j = 1, 2, \ldots, k)$ *be supported on* B. *Then for any* $\gamma < 1$, *there exists a constant* $C_0(\gamma)$ *depending only on* n, α, θ_j, *and* γ, *such that*

$$\int_B \exp\left(n\gamma \left| \frac{L_\theta T_{\Omega,\alpha}(\mathbf{f})(x)}{\|\Omega\|_{n/(n-\alpha)} \prod_{j=1}^k \|f_j\|_{p_j}} \right|^{n/(n-\alpha)} \right) dx \leq C_0(\gamma) R^n,$$

where $L_\theta = \prod_{j=1}^k |\theta_j|^{n/p_j}$ *and*

$$\|\Omega\|_{n/(n-\alpha)} = \left(\int_{S^{n-1}} |\Omega(x')|^{n/(n-\alpha)} d\sigma(x') \right)^{(n-\alpha)/n}.$$

Clearly, if letting $\Omega \equiv 1$ then Theorem 3 is just Theorem 3 in [7]. In proving Theorems 1–3, the k-sublinear maximal operator defined by

$$M_\Omega(\mathbf{f})(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)|\, dy$$

will play a key role. In fact, the proofs of Theorems 1–3 depend heavily on the following theorem for $M_\Omega(\mathbf{f})$.

THEOREM 4. *Let p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$. Then we have the following conclusions.*

(i) *If $p > 1$, $\Omega \in L^s(S^{n-1})$, $s \geq 1$, then $M_\Omega(\mathbf{f})$ maps $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.*

(ii) *If $p = 1$, $\Omega \in L \log^+ L(S^{n-1})$, then $M_\Omega(\mathbf{f})$ maps $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

However, the following theorem will show that the conclusion of Theorem 3 is also valid for $\gamma = 1$.

THEOREM 5. *Suppose that $0 < \alpha < n$, $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, and $p = n/\alpha$ is the harmonic mean of $p_1, p_2, \ldots, p_k > 1$. Then there is a constant $C_0 = C_0(n, \alpha)$ depending only on n and α such that for all $f_j \in L^{p_j}(B)$ with $B = \{x \in \mathbb{R}^n : |x| < R\}$,*

$$\int_B \exp\left(n \left| \frac{L_\theta T_{\Omega,\alpha}(\mathbf{f})(x)}{\|\Omega\|_{n/(n-\alpha)} \prod_{j=1}^k \|f_j\|_{p_j}} \right|^{n/(n-\alpha)} \right) dx \leq C_0 R^n.$$

The proof of Theorem 5 is based on the following result on the exponential integrability of fractional integrals with rough kernel in the endpoint case $p = n/\alpha$.

THEOREM 6. *Let $0 < \alpha < n$ and $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$. Then there is a constant $C_0 = C_0(n, \alpha)$ depending only on n and α such that for all $f \in L^{n/\alpha}(B)$ with $B = \{x \in \mathbb{R}^n : |x| < R\}$,*

$$\frac{1}{|B|} \int_B \exp\left(n \left| \frac{T_{\Omega,\alpha}(f)(x)}{\|\Omega\|_{n/(n-\alpha)} \|f\|_{n/\alpha}} \right|^{n/(n-\alpha)} \right) dx \leq C_0.$$

Theorem 6 is an extension of Theorem 2 in [1], where Adams proved a sharp form of the certain limiting case of the Sobolev embedding theorem by the exponential integrability for the Riesz potential of order α $(0 < \alpha < n)$.

2. SOME LEMMAS

Let us begin by proving a pointwise estimation of $T_{\Omega,\alpha}(\mathbf{f})(x)$.

LEMMA 1. *Suppose that* $0 < \alpha < n, \Omega \in L^s(S^{n-1})$, $s > n/(n-\alpha)$, *and* $1/p_1 + 1/p_2 + \cdots + 1/p_k = 1$. *Then for any* r, $n/(n-\alpha) < r < s$, *there are constants* k_1 *and* k_2, *such that for any* $\delta > 0$ *and any* $\mathbf{f} \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$, *the following equality*

$$|T_{\Omega,\alpha}(\mathbf{f})(x)| \le k_1 \delta^\alpha M_\Omega(\mathbf{f})(x) + k_2 \delta^{\alpha - n/r'} \left(\prod_{j=1}^k \|f_j\|_{p_j} \right)^{1/r'} \left[M_{\Omega^r}(\mathbf{f})(x) \right]^{1/r} \quad (2.1)$$

holds.

Proof. For any $\delta > 0$,

$$\begin{aligned} T_{\Omega,\alpha}(\mathbf{f})(x) &= \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)\,dy \\ &= \int_{|y|<\delta} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)\,dy \\ &\quad + \int_{|y|\ge\delta} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)\,dy \\ &:= I_1 + I_2. \end{aligned}$$

Let us now estimate I_1 and I_2, respectively. Obviously,

$$\begin{aligned} |I_1| &= \left| \sum_{j=0}^\infty \int_{2^{-j-1}\delta \le |y| < 2^{-j}\delta} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)\,dy \right| \\ &\le \sum_{j=0}^\infty (2^{-j-1}\delta)^{\alpha-n} \int_{|y|<2^{-j}\delta} |\Omega(y)|\,|f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\ &\le k_1 \delta^\alpha M_\Omega(\mathbf{f})(x), \end{aligned}$$

where $k_1 = 2^{n-\alpha} \sum_{j=0}^\infty 2^{-j\alpha}$. Let us turn to the estimation of I_2. Since $n/(n-\alpha) < r$, we can choose an $\varepsilon > 0$ such that $(n+\varepsilon)/(n-\alpha) < r$.

Write $n-\alpha=(n+\varepsilon)/r+n/r'-(\alpha+\varepsilon/r)$. By Hölder's inequality, we have

$$\begin{aligned}|I_2| &= \left|\int_{|y|\geq\delta}\frac{\Omega(y)}{|y|^{(n+\varepsilon)/r}}\cdot\frac{1}{|y|^{n/r'-(\alpha+\varepsilon/r)}}\cdot f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\,dy\right|\\
&\leq\left(\int_{|y|\geq\delta}\frac{\Omega(y)}{|y|^{n+\varepsilon}}|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|\,dy\right)^{1/r}\\
&\quad\cdot\left(\int_{|y|\geq\delta}\frac{|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|}{|y|^{n-(\alpha+\varepsilon/r)r'}}dy\right)^{1/r'}\\
&:= J_{21}\cdot J_{22}.\end{aligned}$$

For J_{21}, we have

$$\begin{aligned}J_{21} &= \left(\sum_{j=0}^{\infty}\int_{2^j\delta\leq|y|<2^{j+1}\delta}\frac{|\Omega(y)|r}{|y|^{n+\varepsilon}}|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|\,dy\right)^{1/r}\\
&\leq\left(\sum_{j=0}^{\infty}(2^j\delta)^{-(n+\varepsilon)}\int_{|y|<2^{j+1}\delta}|\Omega(y)|^r|f_1(x-\theta_1 y)|\cdots\right.\\
&\qquad\left.\times|f_k(x-\theta_k y)|\,dy\right)^{1/r}\\
&\leq k_2'\delta^{-\varepsilon/r}\left[M_{\Omega^r}(\mathbf{f})(x)\right]^{1/r},\end{aligned}$$

where $k_2'=(2^n\sum_{j=0}^{\infty}2^{-j\varepsilon})^{1/r}$. On the other hand, by $r>(n+\varepsilon)/(n-\alpha)$, we get $n-(\alpha+\varepsilon/r)r'>0$. Thus, by Hölder's inequality, we have

$$\begin{aligned}J_{22} &\leq\left(\delta^{(\alpha+\varepsilon/r)r'-n}\int_{|y|\geq\delta}|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|dy\right)^{1/r'}\\
&\geq\delta^{\alpha+\varepsilon/r-n/r'}\left[\prod_{j=1}^{k}\left(\int_{|y|\geq\delta}|f_j(x-\theta_j y)|^{p_j}\,dy\right)^{1/p_j}\right]^{1/r'}\\
&\geq L_\theta^{-1/r'}\,\delta^{\alpha+\varepsilon/r-n/r'}\left(\prod_{j=1}^{k}\|f_j\|_{p_j}\right)^{1/r'}.\end{aligned}$$

Thus, by letting $k_2=k_2'L_\theta^{-1/r'}$, we have

$$|I_2|\leq J_{21}\cdot J_{22}\leq k_2\delta^{\alpha-n/r'}\left(\prod_{j=1}^{k}\|f_j\|_{p_j}\right)^{1/r'}\left[M_{\Omega^r}(\mathbf{f})(x)\right]^{1/r}.$$

Therefore

$$|T_{\Omega,\alpha}(\mathbf{f})(x)| \le |I_1| + |I_2| \le k_1 \delta^{\alpha} M_{\Omega}(\mathbf{f})(x) + k_2 \delta^{\alpha - n/r'} \left(\prod_{j=1}^{k} \|f_j\|_{p_j} \right)^{1/r'} \left[M_{\Omega^r}(\mathbf{f})(x) \right]^{1/r}.$$

This is the conclusion of Lemma 1.

LEMMA 2. *Suppose that* $0 < \alpha < n, \Omega \in L^s(S^{n-1})$, $s \ge 1$. *Then there is a* $C > 0$ *depending only on n and* α, *such that*

$$M_{\Omega,\alpha}(\mathbf{f})(x) \le C T_{|\Omega|,\alpha}(|\mathbf{f}|)(x), \tag{2.2}$$

where $|\mathbf{f}| = (|f_1|, \dots, |f_k|)$.

Proof. Denote

$$T_{|\Omega|,\alpha,j}(|\mathbf{f}|)(x) = \int_{2^{j-1} \le |y| < 2^j} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy.$$

Then

$$T_{|\Omega|,\alpha}(|\mathbf{f}|)(x) = \sum_{j \in \mathbb{Z}} T_{|\Omega|,\alpha,j}(|\mathbf{f}|)(x). \tag{2.3}$$

Since

$$\begin{aligned}
&T_{|\Omega|,\alpha,j}(|\mathbf{f}|)(x) \\
&\quad = \int_{2^{j-1} \le |y| < 2^j} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy \\
&\quad \ge 2^{j(\alpha-n)} \int_{2^{j-1} \le |y| < 2^j} |\Omega(y)| \, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy \\
&\quad = 2^{j(\alpha-n)} \bigg[\int_{|y| < 2^j} |\Omega(y)| \, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy \\
&\qquad - \int_{|y| < 2^{j-1}} |\Omega(y)| \, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy \bigg] \\
&\quad = \frac{1}{2^{j(n-\alpha)}} \int_{|y| < 2^j} |\Omega(y)| \, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy \\
&\qquad - \frac{2^{\alpha-n}}{2^{(j-1)(n-\alpha)}} \int_{|y| < 2^{j-1}} |\Omega(y)| \, |f_1(x - \theta_1 y)| \cdots |f_k(x - \theta_k y)| dy,
\end{aligned}$$

we have

$$T_{|\Omega|,\alpha,j}(|\mathbf{f}|)(x) + \frac{2^{\alpha-n}}{2^{(j-1)(n-\alpha)}}\int_{|y|<2^{j-1}}|\Omega(y)|\,|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|dy$$
$$\geq \frac{1}{2^{j(n-\alpha)}}\int_{|y|<2^j}|\Omega(y)|\,|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|dy.$$

Let us take the supremum for $j \in \mathbb{Z}$ on two sides of the above inequality. We get

$$\sup_{j\in\mathbb{Z}} T_{|\Omega|,\alpha,j}(|\mathbf{f}|)(x) \geq (1-2^{\alpha-n})\sup_{j\in\mathbb{Z}}\frac{1}{2^{j(n-\alpha)}} \times\int_{|y|<2^j}|\Omega(y)|\,|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|dy. \tag{2.4}$$

On the other hand, it is easy to prove that

$$M_{\Omega,\alpha}(\mathbf{f})(x) \sim \operatorname*{Sup}_{j\in\mathbb{Z}}\frac{1}{2^{j(n-\alpha)}}\int_{|y|<2^j}|\Omega(y)| \times|f_1(x-\theta_1 y)|\cdots|f_k(x-\theta_k y)|dy. \tag{2.5}$$

Thus, the conclusion of Lemma 2 will follow by observing that the left hand side of (2.4) is bounded by $T_{|\Omega|,\alpha}(|\mathbf{f}|)(x)$.

LEMMA 3. *For any* $0<\varepsilon<\min\{\alpha, n-\alpha\}$, *there is a* $C = C(\varepsilon,\alpha,n)$ *such that*

$$|T_{\Omega,\alpha}(\mathbf{f})(x)| \leq C\left[M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)\right]^{1/2}\left[M_{\Omega,\alpha-\varepsilon}(\mathbf{f})(x)\right]^{1/2}. \tag{2.6}$$

Proof. Given $x\in\mathbb{R}^n$ and $\varepsilon>0$ with $\varepsilon<\min\{\alpha,n-\alpha\}$, we choose δ such that

$$\delta^{2\varepsilon} = M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)/M_{\Omega,\alpha-\varepsilon}(\mathbf{f})(x).$$

Now we put

$$\begin{aligned} T_{\Omega,\alpha}(\mathbf{f})(x) &= \int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^{n-\alpha}}f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\,dy \\ &= \int_{|y|<\delta}\frac{\Omega(y)}{|y|^{n-\alpha}}f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\,dy \\ &\quad + \int_{|y|\geq\delta}\frac{\Omega(y)}{|y|^{n-\alpha}}f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\,dy \\ &:= I_1 + I_2. \end{aligned}$$

Obviously,

$$\begin{aligned}
|I_1| &\le \sum_{j=0}^{\infty} \int_{2^{-j-1}\delta \le |y| < 2^{-j}\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\
&\le \sum_{j=0}^{\infty} (2^{-j-1}\delta)^{\alpha-n} \int_{|y|<2^{-j}\delta} |\Omega(y)|\,|f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\
&\le \sum_{j=0}^{\infty} 2^{n-\alpha} (2^{-j}\delta)^{\varepsilon} M_{\Omega,\,\alpha-\varepsilon}(\mathbf{f})(x) \\
&\le C_1 \delta^{\varepsilon} M_{\Omega,\,\alpha-\varepsilon}(\mathbf{f})(x)
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &\le \sum_{j=1}^{\infty} \int_{2^{j-1}\delta \le |y| < 2^{j}\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\
&\le \sum_{j=1}^{\infty} (2^{j-1}\delta)^{\alpha-n} \int_{|y|<2^{j}\delta} |\Omega(y)|\,|f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\
&\le C_2 \delta^{-\varepsilon} M_{\Omega,\,\alpha+\varepsilon}(\mathbf{f})(x),
\end{aligned}$$

where C_1, C_2 depend only on n, ε, and α. Thus, we have

$$|T_{\Omega,\,\alpha}(\mathbf{f})(x)| \le C\left[\delta^{\varepsilon} M_{\Omega,\,\alpha-\varepsilon}(\mathbf{f})(x) + \delta^{-\varepsilon} M_{\Omega,\,\alpha+\varepsilon}(\mathbf{f})(x)\right]$$

and with the above election of δ, the lemma is proved.

3. BOUNDEDNESS OF $M_{\Omega}(\mathbf{f})(x)$ AND ITS COROLLARIES

In this section we will prove Theorem 4 and its consequences on the weak boundedness of $T_{\Omega,\,\alpha}(\mathbf{f})(x)$ and $M_{\Omega,\,\alpha}(\mathbf{f})(x)$

Proof of Theorem 4. Let us recall that if $k = 1$, $\theta_1 = 1$, and $\Omega \in L^1(S^{n-1})$, then M_{Ω} is a bounded operator on $L^p(\mathbb{R}^n)$ $(p > 1)$ by Calderón and Zygmund [3] and if $k = 1$, $\theta_1 = 1$, and $\Omega \in L\log^+ L(S^{n-1})$, then M_{Ω} is weak type $(1, 1)$ by Christ and Rubio de Francia [5]. Hence the conclusion (i) of Theorem 4 can be easily deduced from Hölder's inequality and the result of Calderón and Zygmund mentioned above. Let us now turn to the proof of (ii). The idea of the proof is taken from [8]. By Hölder's inequality, we have

$$M_{\Omega}(\mathbf{f})(x) \le \prod_{j=1}^{k} \left[M_{\Omega}\left(f_j^{p_j}\right)(x)\right]^{1/p_j}. \tag{3.1}$$

For any $\lambda > 0$, let $\varepsilon_0 = \lambda$, $\varepsilon_k = 1$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1} > 0$ be arbitrary which will be chosen later. Thus, by (3.1) and the above, we get

$$\{x \in \mathbb{R}^n : M_\Omega(\mathbf{f})(x) > \lambda\} \subset \bigcup_{j=1}^{k} \Big\{x \in \mathbb{R}^n : \big[M_\Omega\big(f_j^{p_j}\big)(x)\big]^{1/p_j} > \varepsilon_{j-1}/\varepsilon_j\Big\}. \tag{3.2}$$

Let us now take $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j-1}} = \left[\frac{\prod_{j=1}^{k}\|f_j\| p_j}{\lambda\|f_j^{p_j}\|_1}\right]^{1/p_j}, \qquad j = 1, 2, \ldots, k. \tag{3.3}$$

Therefore, combining (3.2), (3.3) with the weak $(1,1)$ boundedness of M_Ω [5], we get

$$\begin{aligned} \big|\{x : M\Omega(\mathbf{f})(x) > \lambda\}\big| &\le \sum_{j=1}^{k} \Big|\Big\{x : M_\Omega\big(f_j^{p_j}\big)(x) > \big(\varepsilon_{j-1}/\varepsilon_j\big)^{p_j}\Big\}\Big| \\ &\le \sum_{j=1}^{k} C\big(\varepsilon_j/\varepsilon_{j-1}\big)^{p_j}\|f_j^{p_j}\|_1 \\ &\le C/\lambda \prod_{j=1}^{k} \|f_j\|_{p_j}. \end{aligned}$$

This is the conclusion of (ii) in Theorem 4.

From Theorem 4 we can deduce the following weak boundedness of $T_{\Omega,\alpha}(\mathbf{f})(x)$ and $M_{\Omega,\alpha}(\mathbf{f})(x)$

COROLLARY 1. *Suppose that* $0 < \alpha < n, \Omega \in L^s(S^{n-1}), s > n/(n-\alpha)$. *If the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$ *is one, then* $T_{\Omega,\alpha}(\mathbf{f})$ *is a bounded operator from* $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ *to* $L^{n/(n-\alpha),\infty}(\mathbb{R}^n)$.

COROLLARY 2. *Under the conditions as Corollary* 1, *the operator* $M_{\Omega,\alpha}(\mathbf{f})$ *is also bounded from* $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ *to* $L^{n/(n-\alpha),\infty}(\mathbb{R}^n)$.

Proof of Corollary 1. For any $\lambda > 0$ and any $\mathbf{f} = (f_1, \ldots, f_k) \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$, we take $\delta = ((1/\lambda)\prod_{j=1}^{k}\|f_j\|_{p_j})^{1/(n-\alpha)}$. By (2.1), we have

$$\begin{aligned} &\big|\{x : |T_{\Omega,\alpha}(\mathbf{f})(x)| > \lambda\}\big| \\ &\quad\le \big|\{x : k_1\delta^{\alpha}M_\Omega(\mathbf{f})(x) > \lambda/2\}\big| \\ &\qquad + \left|\left\{x : k_2\delta^{\alpha-n/r'}\left(\prod_{j=1}^{k}\|f_j\|_{p_j}\right)^{1/r'}\big[M_{\Omega^r}(\mathbf{f})(x)\big]^{1/r} > \lambda/2\right\}\right| \\ &\quad := A + B. \end{aligned}$$

Let us now give the estimations of A and B as follows. By (ii) of Theorem 4, we have

$$\begin{aligned}
A &= \left|\left\{x : M_\Omega(\mathbf{f})(x) > \frac{\lambda}{2k_1\delta^\alpha}\right\}\right| \\
&\le C_1 \frac{2k_1\delta^\alpha}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j} \\
&= C_1 2k_1 \left(\frac{1}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{\alpha/(n-\alpha)} \cdot \left(\frac{1}{\lambda} \sum_{j=1}^{k} \|f_j\|_{p_j}\right) \\
&= C_1 2k_1 \left(\frac{1}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{n/(n-\alpha)},
\end{aligned}$$

where C_1 depends only on n and s, but not on λ, δ, and $\mathbf{f}$. For B, we get

$$\begin{aligned}
B &= \left|\left\{x : k_2\delta^{\alpha-n/r'}\left(\sum_{j=1}^{k} \|f_j\|_{p_j}\right)^{1/r'} \left[M_{\Omega^r}(\mathbf{f})(x)\right]^{1/r} > \lambda/2\right\}\right| \\
&= \left|\left\{x : M_{\Omega^r}(\mathbf{f})(x) > \left(\frac{\lambda}{2k_2\delta^{\alpha-n/r'}\left(\prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{1/r'}}\right)^r\right\}\right| \\
&\le C_2 \left(\frac{2k_2\delta^{\alpha-n/r'}\left(\prod_{j=1}^{k}\|f_j\|_{p_j}\right)^{1/r'}}{\lambda}\right)^r \left(\prod_{j=1}^{k} \|f_j\|_{p_j}\right),
\end{aligned}$$

where we use again (ii) of Theorem 4, since $\Omega^r \in L^{s/r}(S^{n-1})$ and $s/r > 1$. And C_2 is also independent of λ, δ, and $\mathbf{f}$. So with the election of δ and $r/r' = r - 1$, we have

$$\begin{aligned}
B &\le C_2(2k_2)^r \left(\frac{1}{\lambda}\left(\prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{1/r'}\right)^r \left(\prod_{j=1}^{k} \|f_j\|_{p_j}\right) \cdot \delta^{\alpha r - nr/r'} \\
&\le C_2(2k_2)^r \left(\frac{1}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j}\right)^r \left(\frac{1}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{(\alpha r - nr + n)/(n-\alpha)} \\
&= C_2(2k_2)^r \left(\frac{1}{\lambda} \prod_{j=1}^{k} \|f_j\|_{p_j}\right)^{n/(n-\alpha)}.
\end{aligned}$$

Thus, we complete the proof of Collorary 1.

It is easy to see that the conclusion of Collorary 2 is a direct consequence of Collorary 1 and Lemma 2.

4. PROOFS OF THEOREMS 1 AND 2

We shall alternatively prove Theorems 1 and 2. The proof will be divided into two steps according to the range of p.

Case I. $1 < p < n/\alpha$ (*equivalently* $n/(n-\alpha) < q < \infty$). Let us first prove Theorem 2 in this case. By Collorary 2, $M_{\Omega,\alpha}(\mathbf{f})$ is a bounded operator from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ to $L^{n/(n-\alpha),\infty}(\mathbb{R}^n)$ when $1/p_1 + 1/p_2 + \cdots + 1/p_k = 1$. If we can prove that $M_{\Omega,\alpha}(\mathbf{f})$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ to $L^{\infty}(\mathbb{R}^n)$ with $1/p_1 + 1/p_2 + \cdots + 1/p_k = \alpha/n$, then by the Marcinkiewicz interpolation theorem, $M_{\Omega,\alpha}(\mathbf{f})$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/p_1 + 1/p_2 + \cdots + 1/p_k = 1/p$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Hence it remains to show that $\|M_{\Omega,\alpha}(\mathbf{f})\|_\infty$ is bounded when $\mathbf{f} = (f_1, \dots, f_k) \in L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}(\mathbb{R}^n)$ and $1/p_1 + 1/p_2 + \cdots + 1/p_k = \alpha/n$. Since $s \ge n/(n-\alpha)$ and $\Omega \in L^s(S^{n-1})$, by Hölder's inequality, we have

$$
\begin{aligned}
M_{\Omega,\alpha}(\mathbf{f})(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)|\,|f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\,dy \\
&\le \sup_{r>0} \frac{1}{r^{n-\alpha}} \left(\int_{|y|<r} |\Omega(y)|^{n/(n-\alpha)} dy \right)^{(n-\alpha)/n} \\
&\qquad \cdot \left(\int_{|y|<r} |f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)|^{n/\alpha} dy \right)^{\alpha/n} \\
&\le \|\Omega\|_{n/(n-\alpha)} \\
&\qquad \times \sup_{r>0} \left(\int_{|y|<r} |f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)|^{n/\alpha} dy \right)^{\alpha/n}.
\end{aligned}
$$

Let $l_j = p_j \alpha/n$. Since $1/p_1 + 1/p_2 + \cdots + 1/p_k = \alpha/n$, we have $1/l_1 + 1/l_2 + \cdots + 1/l_k = 1$. Using Hölder's inequality for $(l_1, l_2, \dots, l_k)$, we get

$$
\begin{aligned}
&\left(\int_{|y|<r} |f_1(x-\theta_1 y) \cdots f_k(x-\theta_k y)|^{n/\alpha}\, dy \right)^{\alpha/n} \\
&\quad \le \prod_{j=1}^{k} \left(\int_{|y|<r} |f_j(x-\theta_j y)|^{l_j n/\alpha}\, dy \right)^{\alpha/l_j n} \\
&\quad = \prod_{j=1}^{k} \left(\int_{|y|<r} |f_j(x-\theta_j y)|^{p_j}\, dy \right)^{1/p_j} \\
&\quad \le C_\theta \prod_{j=1}^{k} \|f_j\|_{p_j}.
\end{aligned}
$$

Therefore, we have

$$M_{\Omega,\alpha}(\mathbf{f})(x) \le C_\theta \|\Omega\|_{n/(n-\alpha)} \prod_{j=1}^{k} \|f_j\|_{p_j}. \tag{4.1}$$

This confirms our claim. Thus, the conclusion of Theorem 2 in Case I holds. Now we turn to the proof of Theorem 1 in Case I. Since $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, we can choose $\varepsilon > 0$ with $\varepsilon < \min\{\alpha, n-\alpha\}$ such that

$$1/q - \varepsilon/n > 0 \qquad \text{and} \qquad 1/q + \varepsilon/n < 1.$$

By letting $1/q_1 = 1/q - \varepsilon/n$ and $1/q_2 = 1/q + \varepsilon/n$, we have

$$1/q_1 = 1/p - (\alpha+\varepsilon)/n \qquad \text{and} \qquad 1/q_2 = 1/p - (\alpha-\varepsilon)/n.$$

From the conclusion of Theorem 2 in Case I, it follows that

$$\|M_{\Omega,\alpha+\varepsilon}(\mathbf{f})\|_{q_1} \le C \prod_{j=1}^{k} \|f_j\|_{p_j} \tag{4.2}$$

and

$$\|M_{\Omega,\alpha+\varepsilon}(\mathbf{f})\|_{q_1} \le C \prod_{j=1}^{k} \|f_j\|_{p_j} \tag{4.3}$$

Now if we denote $l_1 = 2q_1/q$ and $l_2 = 2q_2/q$, then it is easy to check that $l_1, l_2 > 1$ and $1/l_1 + 1/l_2 = 1$. Thus, by (2.6) and Hölder's inequality for l_1 and l_2, we get

$$\begin{aligned}
\|T_{\Omega,\alpha}(\mathbf{f})\|_q &\le C\left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)\right]^{q/2} \left[M_{\Omega,\alpha-\varepsilon}(\mathbf{f})(x)\right]^{q/2} dx\right)^{1/q} \\
&\le C\left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)\right]^{l_1 q/2} dx\right)^{1/l_1 q} \\
&\quad \times \left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha-\varepsilon}(\mathbf{f})(x)\right]^{l_2 q/2} dx\right)^{1/l_2 q} \\
&= C\left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)\right]^{q_1} dx\right)^{1/2q_1} \\
&\quad \times \left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha+\varepsilon}(\mathbf{f})(x)\right]^{q_2} dx\right)^{1/2q_2} \\
&\le C \prod_{j=1}^{k} \|f_j\|_{p_j},
\end{aligned}$$

where we use (4.2) and (4.3) in the last step above. This completes the proof of Theorem 1 in Case I.

Case II. $n/(n+\alpha) \le p \le 1$ (*equivalently* $1 \le q \le n/(n-\alpha)$).

In this case we first give the proof of Theorem 1 by an induction on k. Assume that $k = 2$ and $p_1 \ge p_2 > 1$. Let us first consider the case of $q = 1$. Since $p < n/\alpha$, we have $p_2 < n/\alpha$. For $q = 1$, we have

$$\begin{aligned}
\|T_{\Omega,\alpha}(\mathbf{f})\|_1 &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)|\,|f_2(x-\theta_2 y)|\,dy\,dx \\
&= \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} |f_1(x-\theta_1 y)|\,|f_2(x-\theta_2 y)|\,dx\,dy \\
&= \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} \int_{\mathbb{R}^n} |f_1(x)|\,|f_2(x-(\theta_2-\theta_1)y)|\,dx\,dy \\
&= \int_{\mathbb{R}^n} |f_1(x)| \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_2(x-(\theta_2-\theta_2)y)|\,dy\,dx \\
&= |\theta_2-\theta_1|^{-\alpha} \int_{\mathbb{R}^n} |f_1(x)| \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_2(x-y)|\,dy\,dx \\
&= C(\theta_1,\theta_2,\alpha) \int_{\mathbb{R}^n} |f_1(x)| T_{|\Omega|,\alpha}(|f_2|)(x)\,dx \\
&\le C(\theta_1,\theta_2,\alpha) \|f_2\|_{p_1} \|T_{|\Omega|,\alpha}(|f_2|)\|_{p_1'}.
\end{aligned}$$

Note that $q = 1$ implies $1/p_1' = 1/p_2 - \alpha/n$. Since $1 < p_2 < n/\alpha$ and $\Omega \in L^s(S^{n-1})$, $s > n/(n-\alpha)$, we may apply (L^p, L^q)-boundedness for $T_{\Omega,\alpha}$ to obtain $\|T_{|\Omega|,\alpha}(|f_2|)\|_{p_2}$. Thus, we get

$$\|T_{\Omega,\alpha}(\mathbf{f})\|_1 \le C_{\theta,\alpha} \prod_{j=1}^{2} \|f_j\|_{p_j}. \tag{4.4}$$

For the general case $1 < q \le n/(n-\alpha)$, the conclusion follows from the multilinear interpolation theorem between (4.4) and the conclusion of Theorem 1 in Case I.

Now, suppose that the conclusion of Theorem 1 in Case II is true for $k-1$, $k \ge 3$. We shall show that it is also true for k. Again we first consider the case $q = 1$. We may assume without loss of generality that $p_1 \geq p_2 \geq \cdots \geq p_k > 1$. Now,

$$\begin{aligned}
\|T_{\Omega,\alpha}(\mathbf{f})\|_1 &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)| \cdots |f_k(x-\theta_k y)|\, dy\, dx \\
&= \int_{\mathbb{R}^n} |f_1(x)| \int_{\mathbb{R}^n} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_2(x-\xi_2 y)| \cdots |f_k(x-\xi_k y)|\, dy\, dx,
\end{aligned}$$

where $\xi_j = \theta_j - \theta_1$, $j = 2, 3, \ldots, k$ and mutually distinct. Thus, we have

$$\|T_{\Omega,\alpha}(\mathbf{f})\|_1 \leq \|f_1\|_{p_1} \cdot \|T_{|\Omega|,\alpha}(|f_2|, \ldots, |f_k|)\|_{p_1'}. \tag{4.5}$$

Denote q_1 by $1/q_1 = 1/p - 1/p_1$, then we have the following conclusions:

$$\begin{aligned}
&\text{(i)} \quad 1/q_1 = 1/p_2 + 1/p_3 + \cdots + 1/p_k. \\
&\text{(ii)} \quad 1/p_1' = 1/q_1 - \alpha/n. \\
&\text{(iii)} \quad n/(n+\alpha) \leq q_1 < n/\alpha.
\end{aligned} \tag{4.6}$$

In fact, (i) is obvious by the choice of q_1. For (ii), from $q = 1$ it follows

$$1 = 1/p - \alpha/n = 1/q_1 + 1/p_1 - \alpha/n,$$

i.e.,

$$1/p_1' = 1/q_1 - \alpha/n.$$

Moreover, we can see that (iii) is deduced from (ii) and the fact $1/q_1 < 1/p = (n+\alpha)/n$. Thus, if $1 < q_1 < n/\alpha$, then by (i), (ii) of (4.6) and the result of Theorem 1 in Case I, we see that

$$\|T_{|\Omega|,\alpha}(|f_2|, \ldots, |f_k|)\|_{p_1'} \leq C_\xi \prod_{j=2}^{k} \|f_j\|_{p_j}. \tag{4.7}$$

If $n/(n+\alpha) \leq q \leq 1$, by (i), (ii) of (4.6) and the induction hypothesis, we get also

$$\|T_{|\Omega|,\alpha}(|f_2|, \ldots, |f_k|)\|_{p_1'} \leq C_\xi \prod_{j=2}^{k} \|f_j\|_{p_j}. \tag{4.8}$$

Combining (4.5), (4.7) with (4.8), we obtain

$$\|T_{\Omega,\alpha}(\mathbf{f})\|_1 \le C_\theta \prod_{j=1}^{k} \|f_j\|_{p_j}. \tag{4.9}$$

Using again the multilinear interpolation theorem between (4.9) and the conclusion of Theorem 1 in Case I, we get

$$\|T_{\Omega,\alpha}(\mathbf{f})\|_q \le C_\theta \prod_{j=1}^{k} \|f_j\|_{p_j} \tag{4.10}$$

for $n/(n+\alpha) \le p \le 1$ (equivalently $1 \le q \le n/(n-\alpha)$). This finishes the proof of Theorem 1 in Case II. By this and Lemma 2, we can immediately obtain the conclusion of Theorem 2 in this case.

5. PROOF OF THEOREM 3

Let us first assume that $\|f_j\|_{p_j} = 1, j = 1, 2, \ldots, k$. From the process of proving Lemma 1 we see that for any $\delta > 0$ and $x \in B$,

$$\begin{aligned} |T_{\Omega,\alpha}(\mathbf{f})(x)| &\le k_1 \delta^\alpha M_\Omega(\mathbf{f})(x) \\ &\quad + \int_{|y|\ge\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)|\, dy. \end{aligned} \tag{5.1}$$

Since all f_j are supported on the ball B and $x \in B$, we have

$$\begin{aligned} &\int_{|y|\ge\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)|\, dy \\ &\quad = \int_{\delta\le|y|\le 2\sigma R} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)|\, dy, \end{aligned} \tag{5.2}$$

where $\sigma = \min\{1/|\theta_1|, 1/|\theta_2|, \ldots, 1/|\theta_k|\}$. Thus, by (5.2) and Hölder's inequality with the exponents $p_1, p_2, \ldots, p_k$ and $n/(n-\alpha)$, we get

$$\begin{aligned}
&\int_{|y|\geq\delta} \frac{|\Omega(y)|}{|y|^{n-\alpha}} |f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)|\,dy \\
&\quad\leq \left(\int_{\delta\leq|y|\leq 2\sigma R} \left(\frac{|\Omega(y)|}{|y|^{n-\alpha}}\right)^{n/(n-\alpha)} dy\right)^{(n-\alpha)/n} \\
&\qquad\cdot\left(\int_{\delta\leq|y|\leq 2\sigma R} |f_1(x-\theta_1 y)|^{p_1}\,dy\right)^{1/p_1}\cdots \\
&\qquad\times\left(\int_{\delta\leq|y|\leq 2\sigma R} |f_k(x-\theta_k y)|^{p_k}\,dy\right)^{1/p_k} \\
&\quad\leq L_\theta^{-1}\cdot\left(\int_{\delta\leq|y|\leq 2\sigma R} |\Omega(y)|^{n/(n-\alpha)}\cdot|y|^{-n}\,dy\right)^{(n-\alpha)/n} \\
&\quad= L_\theta^{-1}\cdot\left(\int_{S^{n-1}} |\Omega(y')|^{n/(n-\alpha)}\int_\delta^{2\sigma R}\frac{dr}{r}\,d\sigma(y')\right)^{(n-\alpha)/n} \\
&\quad= L_\theta^{-1}\cdot\|\Omega\|_{n/(n-\alpha)}\cdot\left(\log\frac{2\sigma R}{\delta}\right)^{(n-\alpha)/n},
\end{aligned}$$

where $L_\theta = \prod_{j=1}^{k}|\theta_j|^{n/p_j}$. Thus, by (5.1) and the above, we obtain

$$|T_{\Omega,\alpha}(\mathbf{f})(x)| \leq k_1\delta^\alpha M_\Omega(\mathbf{f})(x) + L_\theta^{-1}\cdot\|\Omega\|_{n/(n-\alpha)}\cdot\left(\log\frac{2\sigma R}{\delta}\right)^{(n-\alpha)/n}, \tag{5.3}$$

provided $x\in B$ and $0<\delta\leq 2\sigma R$. In particular, the choice of $\delta = 2\sigma R$ yields $|T_{\Omega,\alpha}(\mathbf{f})(x)| \leq k_1(2\sigma R)^\alpha M_\Omega(\mathbf{f})(x)$ for all $x\in B$ and therefore the election of

$$\delta = \delta(x) = \varepsilon\left[|T_{\Omega,\alpha}(\mathbf{f})(x)|/k_1 M_\Omega(\mathbf{f})(x)\right]^{1/\alpha}$$

will satisfy $\delta\leq 2\sigma R$ for all $\varepsilon\leq 1$. Now, (5.3) implies

$$\begin{aligned}
|T_{\Omega,\alpha}(\mathbf{f})(x)| &\leq \varepsilon^\alpha |T_{\Omega,\alpha}(\mathbf{f})(x)| \\
&\quad + A\cdot\left[\frac{1}{n}\log\left(\frac{2\sigma R\left[k_1 M_\Omega(\mathbf{f})(x)\right]^{1/\alpha}}{\varepsilon|T_\Omega,\alpha(\mathbf{f})(x)|^{1/\alpha}}\right)^n\right]^{(n-\alpha)/n},
\end{aligned} \tag{5.4}$$

where $A = L_\theta^{-1} \cdot \|\Omega\|_{n/(n-\alpha)}$. If we denote $\gamma = (1 - \varepsilon^\alpha)^{n/(n-\alpha)}$, then (5.4) is equivalent to

$$\frac{n\gamma}{A^{n/(n-\alpha)}} |T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/(n-\alpha)} \le \log\left(\frac{\eta [M_\Omega(\mathbf{f})(x)]^{n/\alpha}}{|T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/\alpha}} \right), \qquad (5.5)$$

where $\eta = (2\sigma R)^n \varepsilon^{-n} k_1^{n/\alpha}$. By exponentiating (5.5), we get

$$\exp\left(\frac{n\gamma}{A^{n/(n-\alpha)}} |T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/(n-\alpha)} \right) \le \frac{\eta [M_\Omega(\mathbf{f})(x)]^{n/\alpha}}{|T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/\alpha}}. \qquad (5.6)$$

Let $B_1 = \{x \in B : |T_{\Omega,\alpha}(\mathbf{f})(x)| \ge 1\}$ and $B_2 = B \setminus B_1$. By (5.6) and (i) of Theorem 4, we have

$$\begin{aligned} \int_{B_1} \exp\left(\frac{n\gamma}{A^{n/(n-\alpha)}} |T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/(n-\alpha)} \right) dx &\le \eta \int_{B_1} [M_\Omega(\mathbf{f})(x)]^{n/\alpha}\, dx \\ &\le C_1 \eta = C_2 \varepsilon^{-n} R^n, \end{aligned}$$

where $C_2 = C_1 k_1^{n/\alpha} (2\sigma)^n$. On the other hand,

$$\begin{aligned} \int_{B_2} \exp\left(\frac{n\gamma}{A^{n/(n-\alpha)}} |T_{\Omega,\alpha}(\mathbf{f})(x)|^{n/(n-\alpha)} \right) dx &\le \exp\left(\frac{n}{A^{n/(n-\alpha)}} \right) \cdot |B_2| \\ &\le C_3 \omega_n R^n = C_4 R^n, \end{aligned}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Thus, adding the integrals above over B_1 and B_2, we have

$$\int_B \exp\left(n\gamma \left| \frac{T_{\Omega,\alpha}(\mathbf{f})(x)}{A} \right|^{n/(n-\alpha)} \right) dx \le C_0(\gamma) R^n, \qquad (5.7)$$

where $C_0(\gamma) = \max\{C_2, C_4\}(1 + (1 - \gamma^{(n-\alpha)/n})^{-n/\alpha})$. Let us now turn to the general $\mathbf{f}$. If $\|f_j\|_{p_j} \ne 1\,(j = 1, 2, \ldots, k)$, then we denote $g_j = f_j / \|f_j\|_{p_j}$ and $\mathbf{g} = (g_1, \ldots, g_k)$. Obviously, we have

$$|T_{\Omega,\alpha}(\mathbf{g})(x)| = |T_{\Omega,\alpha}(\mathbf{f})(x)| \Big/ \prod_{j=1}^{k} \|f_j\|_{p_j}. \qquad (5.8)$$

Combining (5.7) with (5.8), we get

$$\int_B \exp\left(n\gamma \left| \frac{T_{\Omega,\alpha}(\mathbf{f})(x)}{A\prod_{j=1}^{k}\|f_j\|_{p_j}} \right|^{n/(n-\alpha)} \right) dx \le C_0(\gamma) R^n,$$

which is just our assertion.

6. PROOFS OF THEOREMS 5 AND 6

Like the proof of Theorem 1 and [2], Theorem 5 is actually a direct consequence of Theorem 6 by using a multilinear interpolation theorem for the Orlicz spaces (with the Luxember norm) given in [2]. Therefore, it will suffice to prove Theorem 6.

As usual we denote the distribution function of f by $\lambda(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}| (s > 0)$, the nonincreasing rearrangement of f by $f^*(t) = \inf\{s : \lambda(x) \le t\} (t > 0)$ and $f^{**}(t) = (1/t)\int_0^t f^*(s) ds \, (t > 0)$.

Lemma 4. *Suppose that* $0 < \alpha < n, \Omega \in L^{n/(n-\alpha)}(S^{n-1})$, and $K(x) = \Omega(x)/|x|^{n-\alpha}$. *Then we have*

$$K^*(t) = \left(\frac{A}{nt}\right)^{(n-\alpha)/n} \tag{6.1}$$

and

$$K^{**}(t) = \frac{n}{\alpha} K^*(t), \tag{6.2}$$

where $A = \|\Omega\|_{n/(n-\alpha)}^{n/(n-\alpha)}$.

The conclusion of Lemma 4 can be easily deduced by a direct computation. We omit the detail here.

Lemma 5. (Adams [1]). *Let* $a(s,t)$ *be a nonnegative measurable function on* $(-\infty, +\infty) \times [0, +\infty)$ *such that*

$$a(s,t) \le 1, \qquad a.e., \text{ if } 0 < s < t, \tag{6.3}$$

$$\operatorname*{Sup}_{t>0} \left(\int_{-\infty}^{0} + \int_t^{\infty} a(s,t)^{p'} \, ds \right)^{1/p'} = b < \infty. \tag{6.4}$$

Then there is a constant $C_0 = C_0(p, b)$ *such that for* $\phi \ge 0$ *with*

$$\int_{-\infty}^{\infty} \phi(s)^p ds \le 1, \tag{6.5}$$

we have

$$\int_0^\infty e^{-F(t)}dt \le C_0, \tag{6.6}$$

where

$$F(t) = t - \left(\int_{-\infty}^\infty a(s,t)\phi(s)ds\right)^{p'}. \tag{6.7}$$

Let us now turn to prove Theorem 6. We should point out that the idea of the proof here is the same as the one of Theorem 2 in [1]. Let us first assume that $\|f\|_{n/\alpha} = 1$. Using O'Neil's lemma (Lemma 1.5 in [10]) for the rearrangement of a convolution, we have

$$\begin{aligned}(T_{\Omega,\alpha}(f))^*(t) &\le (T_{\Omega,\alpha}(f))^{**}(t)\\ &\le tf^{**}(t)K^{**}(t) + \int_t^\infty f^*(s)K^*(s)ds\\ &= \left(\frac{A}{n}\right)^{1/p'}\left(pt^{-1/p'}\int_0^t f^*(s)\,ds + \int_t^{|B|} f^*(s)s^{-1/p'}\,ds\right),\end{aligned} \tag{6.8}$$

where and below $p = n/\alpha$, $p' = n/(n-\alpha)$.

Let

$$a(s,t) = \begin{cases} 1, & \text{for } 0 < s < t,\\ pe^{(t-s)/p'}, & \text{for } t < s < \infty,\\ 0, & \text{for } -\infty < s \le 0,\end{cases}$$

and

$$\phi(x) = |B|^{1/p} f^*(|B|e^{-s})e^{-s/p}.$$

Then we have

$$\operatorname*{Sup}_{t>0}\left(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{p'}\,ds\right)^{1/p'} = p < \infty$$

and

$$\int_{-\infty}^\infty \phi(s)^p ds + \int_0^{|B|} f^*(s)^p ds = \int_B |f(x)|^p dx \le 1.$$

Thus, $a(s,t)$ and $\phi(s)$ satisfy (6.3)–(6.5). By Lemma 5, there is a constant C_0 depending only on p such that

$$\int_0^\infty e^{-F(t)}\,dt \le C_0, \tag{6.9}$$

where

$$F(t) = t - \left(\int_{-\infty}^{\infty} a(s,t)\phi(s)\,ds\right)^{p'}.$$

On the other hand, from the definitions of $a(s,t)$ and $\phi(s)$, it follows that

$$F(t) = t - \left\{\int_0^t |B|^{1/p} f^*(|B|e^{-s})e^{-s/p}\,ds \right.$$
$$\left. + \int_t^{\infty} pe^{(t-s)/p'}|B|^{1/p} f^*(|B|e^{-s})e^{-s/p} ds\right\}^{p'}.$$

By the change of variables, we have

$$F\left(\log\frac{|B|}{t}\right) = \log\frac{|B|}{t} - \left\{pt^{-1/p'}\int_0^t f^*(s)\,ds + \int_t^{|B|} f^*(s)s^{-1/p'}\,ds\right\}^{p'} \tag{6.10}$$

Combining (6.8), (6.9) with (6.10), we get

$$\begin{aligned}
C_0 &\geq \int_0^{\infty} e^{-F(t)}dt = \int_0^{|B|} t^{-1}e^{-F(\log(|B|/t))}\,dt \\
&= \int_0^{|B|} t^{-1}\exp\left\{\left(pt^{-1/p'}\int_0^t f^*(s)\,ds + \int_0^{|B|} f^*(s)s^{-1/p'}\,ds\right)^{p'} - \log\frac{|B|}{t}\right\}dt \\
&= \frac{1}{|B|}\int_0^{|B|} \exp\left\{\left(pt^{-1/p'}\int_0^t f^*(s)\,ds + \int_0^{|B|} f^*(s)s^{-1/p'}\,ds\right)^{p'}\right\}dt \\
&\geq \frac{1}{|B|}\int_0^{|B|} \exp\left\{\frac{n}{A}(T_{\Omega,\alpha}(f))^*(t)^{p'}\right\}dt \\
&= \frac{1}{|B|}\int_{|B|} \exp\left\{\frac{n}{A}|T_{\Omega,\alpha}(f)(x)|^{p'}\right\}dx,
\end{aligned}$$

i.e.,

$$\frac{1}{|B|}\int_{|B|} \exp\left\{n\left|\frac{T_{\Omega,\alpha}(f)(x)}{\|\Omega\|_{n/(n-\alpha)}}\right|^{n/(n-\alpha)}\right\}dx \leq C_0, \tag{6.11}$$

where $\|f\|_{n/\alpha} = 1$.

Now let us consider the general case. If $\|f\|_{n/\alpha} \neq 1$, then we denote $g = f/\|f\|_{n/\alpha}$. Thus,

$$|T_{\Omega,\alpha}(g)(x)| = |T_{\Omega,\alpha}(f)(x)|/\|f\|_{n/\alpha}$$

and $\|g\|_{n/\alpha} = 1$. From (6.11), it follows that

$$\frac{1}{|B|}\int_{|B|} \exp\left\{ n \left| \frac{T_{\Omega,\alpha}(f)(x)}{\|\Omega\|_{n/(n-\alpha)}\|f\|_{n/\alpha}} \right|^{n/(n-\alpha)} \right\} dx \leq C_0 .$$

This finishes the proof of Theorem 6.

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